HIGH DENSITY PIECEWISE SYNDETICITY OF PRODUCT SETS IN AMENABLE GROUPS

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Abstract. M. Beiglböck, V. Bergelson, and A. Fish proved that if G is a countable amenable group and A and B are subsets of G with positive Banach density, then the product set AB is piecewise syndetic. This means that there is a finite subset E of G such that EAB is thick, that is, EAB contains translates of any finite subset of G. When $G = \mathbb{Z}$, this was first proven by R. Jin. We prove a quantitative version of the aforementioned result by providing a lower bound on the density (with respect to a Følner sequence) of the set of witnesses to the thickness of EAB. When $G = \mathbb{Z}^d$, this result was first proven by the current set of authors using completely different techniques.

1. Introduction

In the paper [6], R. Jin proved that if A and B are subsets of $\mathbb{Z}$ with positive Banach density, then $A+B$ is *piecewise syndetic*. This means that there is $m \in \mathbb{N}$ such that $A+B+[-m,m]$ is *thick*, i.e. it contains arbitrarily large intervals. Jin's result has since been extended in two different ways. First, using ergodic theory, M. Beiglböck, V. Bergelson, and A. Fish [2] established Jin's result for arbitrary countable amenable groups (with suitable notions of Banach density and piecewise syndeticity); in [5], M. Di Nasso and M. Lupini gave a simpler proof of the amenable group version of Jin's theorem using nonstandard analysis which works for arbitrary (not necessarily countable) amenable groups and also gives a bound on the size of the finite set needed to establish that the product set is thick. Second, in [4] the current set of authors established a "quantitative version" of Jin's theorem by proving that there is $m \in \mathbb{N}$ such that the set of witnesses to the thickness of $A + B + [-m, m]$ has upper density at least as large as the upper density of A. (We actually prove this result for subsets of $\mathbb{Z}^d$ for any d.) The goal of this article is to prove the quantitative version of the result of Beiglböck, Bergelson, and Fish.

In the following, we assume that G is a countable amenable group[1]: for every finite subset E of G and every $\varepsilon > 0$, there exists a finite subset L of G such that, for every $x \in E$, we have $|xL \bigtriangleup L| \leq \varepsilon |L|$. We recall that G is amenable if and only if it has a (left) *Følner sequence*, which is a sequence $\mathcal{S} = (S_n)$ of finite subsets of G such that, for every $x \in G$, we have $\frac{|xS_n \triangle S_n|}{|S_n|} \to 0$ as $n \to +\infty$. A *two-sided* Følner sequence (S_n) satisfies a stronger requirement: for every $x, y \in G$, we have $\frac{|xS_n y \triangle S_n|}{|S_n|} \to 0$ as $n \to +\infty$. Every countable amenable group has a two-sided Følner sequence [8].

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[1]For those not familiar with amenable groups, let us mention in passing that the class of (countable) amenable groups is quite robust, e.g. contains all finite and abelian groups and is closed under subgroups, quotients, extensions, and direct limits. It follows, for example, that every countable virtually solvable group is amenable.

If $\mathcal{S}$ is a Følner sequence and $A \subseteq G$, we define the corresponding (upper) *$\mathcal{S}$-density* of A to be

$$d_{\mathcal{S}}(A) := \limsup_{n \to +\infty} \frac{|A \cap S_n|}{|S_n|}.$$

For example, note that if $G = \mathbb{Z}^d$ and $\mathcal{S} = (S_n)$ where $S_n = [-n, n]^d$, then $d_{\mathcal{S}}$ is the usual notion of upper density for subsets of $\mathbb{Z}^d$.

Following [2], we define the (left) *Banach density of* A, $\mathrm{BD}(A)$, to be the supremum of $d_{\mathcal{S}}(A)$ where $\mathcal{S}$ ranges over all Følner sequences of G. One can verify (see [2]) that this notion of Banach density agrees with the usual notion of Banach density when $G = \mathbb{Z}^d$. In [3, Definition 2.11], the authors consider the *two-sided version* $\mathrm{BD}_2(A)$ of Banach density, which is defined as the supremum of $d_{\mathcal{S}}(A)$ where $\mathcal{S}$ ranges over all *two-sided* Følner sequences of G. Clearly BD_2 is bounded from above by BD with strict inequality possible ([3, Example 4.12]). Note that in the case of abelian groups, the two densities coincide.

Recall that a subset A of G is *thick* if for every finite subset L of G there exists a right translate Lx of L contained in A. A subset A of G is *piecewise syndetic* if FA is thick for some finite subset F of G.

Definition. Suppose that G is a countable discrete group, $\mathcal{S}$ is a Følner sequence for G, A is a subset of G, and $\alpha > 0$. We say that A is

- *$\mathcal{S}$-thick of level* α if for every finite subset L of G, the set $\{x \in G : Lx \subseteq A\}$ has $\mathcal{S}$-density at least α;
- *$\mathcal{S}$-syndetic of level* α if there exists a finite subset F of G such that FA is $\mathcal{S}$-thick of level α.

The following is the main result of this paper:

Theorem. *Suppose that G is a countable amenable group and $\mathcal{S}$ is a Følner sequence for G. If A and B are subsets of G such that $d_{\mathcal{S}}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$, then BA is $\mathcal{S}$-syndetic of level α' for every $\alpha' < \alpha$. If moreover $\mathrm{BD}_2(B) > 0$, then BA is $\mathcal{S}$-syndetic of level α.*

When $G = \mathbb{Z}^d$ and $\mathcal{S} = (S_n)$ where $S_n = [-n, n]^d$, we recover [4, Theorem 14]. We also recover [4, Theorem 18], which states (in the current terminology) that if $A, B \subseteq \mathbb{Z}^d$ are such that $\underline{d}(A) = \alpha > 0$ and $\mathrm{BD}(B) > 0$, then $A + B$ is $\mathcal{S}'$-syndetic of level α for any subsequence $\mathcal{S}'$ of the aforementioned $\mathcal{S}$.

As in the proof of [4, Theorem 14], we deduce the second assertion in the main theorem from the first one. In fact, if $d_{\mathcal{S}}(A) = \alpha$, then either A is already $\mathcal{S}$-thick of level α or else there is $g \in G$ for which $d_{\mathcal{S}}(A \cup gA) > \alpha$. We then show that there exists a subset B_0 of B of positive Banach density such that B_0A and B_0gA are both contained in BA, whence we can apply the first assertion to B_0 and $A \cup gA$.

In their proof of the amenable group version of Jin's theorem, the authors of [5] give a bound on the size of a finite set needed to witness piecewise syndeticity: if G is a countable amenable group and A and B are subsets of G of Banach densities α and β respectively, then there is a finite subset E of G with $|E| \leq \frac{1}{\alpha\beta}$ such that EAB is thick. In Section 3, we improve upon this theorem in two ways: we slightly improve the bound on $|E|$ from $\frac{1}{\alpha\beta}$ to $\frac{1}{\alpha\beta} - \frac{1}{\alpha} + 1$ and we show that, if $\mathcal{S}$ is any Følner sequence such that $d_{\mathcal{S}}(A) > 0$, then EAB is $\mathcal{S}$-thick of level s for some $s > 0$.

It is interesting to note that it is not clear how to use the techniques of this paper to generalize the quantitative results concerning lower density [4, Theorems 19 and 22] to the amenable group setting.

Notions from nonstandard analysis. We use nonstandard analysis to prove our main results. An introduction to nonstandard analysis with an eye towards applications to combinatorics can be found in [7]. Here, we just fix notation.

If r, s are finite hyperreal numbers, we write $r \lesssim s$ to mean $\mathrm{st}(r) \leq \mathrm{st}(s)$ and we write $s \approx r$ to mean $\mathrm{st}(r) = \mathrm{st}(s)$.

If X is a hyperfinite subset of ${}^*G$, we denote by μ_X the corresponding Loeb measure. If, moreover, $Y \subseteq {}^*G$ is internal, we abuse notation and write $\mu_X(Y)$ for $\mu_X(X \cap Y)$.

If $\mathcal{S} = (S_n)$ is a Følner sequence for G and ν is an infinite hypernatural number, then we denote by S_ν the value at ν of the nonstandard extension of $\mathcal{S}$. It follows readily from the definition that $d_{\mathcal{S}}(A)$ is the maximum of $\mu_{S_\nu}({}^*A)$ as ν ranges over all infinite hypernatural numbers. It is also not difficult to verify that a countable discrete group G is amenable if and only if it admits a *Følner approximation*, which is a hyperfinite subset X of ${}^*G$ such that $|gX \bigtriangleup X| / |X|$ is infinitesimal for every $g \in G$. (One direction of this equivalence is immediate: if (S_n) is a Følner sequence for G, then S_ν is a Følner approximation for G whenever ν is an infinite hypernatural number.) The Banach density $\mathrm{BD}(A)$ of an infinite subset A of G is then the maximum of $\mu_X({}^*A)$ as X ranges over all Følner approximations X of G; see [5].

2. High density piecewise syndeticity

In this section, we fix a countable amenable group G and an arbitrary Følner sequence $\mathcal{S} = (S_n)$ for G.

Lemma 2.1. *If $d_{\mathcal{S}}(A) \geq \alpha$ and $\mathrm{BD}(B) \geq \beta$, then there exist a Følner approximation Y of G and an infinite hypernatural number ν such that*

$$\frac{|{}^*A \cap S_\nu|}{|S_\nu|} \gtrsim \alpha, \quad \frac{|{}^*B \cap Y|}{|Y|} \gtrsim \beta$$

and

$$\frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{\left|x\left({}^*A \cap S_\nu\right)^{-1} \cap \left({}^*B \cap Y\right)\right|}{|Y|} \gtrsim \alpha\beta.$$

Proof. Pick a Følner approximation Y of G such that

$$\frac{|{}^*B \cap Y|}{|Y|} \gtrsim \beta.$$

We claim that there exists an infinite hypernatural number ν such that

$$\frac{|{}^*A \cap S_\nu|}{|S_\nu|} \gtrsim \alpha \quad \text{and} \quad \sum_{y \in Y} \frac{\left|y^{-1}S_\nu \bigtriangleup S_\nu\right|}{|S_\nu|} \approx 0.$$

This can be seen by applying transfer to the statement "for every finite subset E of G and for every natural number n_0 there exists $n > n_0$ such that

$$\frac{1}{|S_n|}\left|A \cap S_n\right| > \alpha - 2^{-n_0} \quad \text{and} \quad \frac{1}{|S_n|} \sum_{x \in E} \left|x^{-1}S_n \bigtriangleup S_n\right| < 2^{-n_0}\text{''},$$

and then applying the transferred statement to Y and an infinite hypernatural number ν_0. Set $C = {}^*A \cap S_\nu$, and $D = {}^*B \cap Y$.

We finish by arguing as in the proof of [5, Lemma 2.3]. For the sake of completeness, we include the details here. Let χ_C denote the characteristic function of C. We then have

$$\begin{aligned}\frac{1}{|S_\nu|}\sum_{x\in S_\nu}\frac{\left|xC^{-1}\cap D\right|}{|Y|} &= \frac{1}{|S_\nu|}\sum_{x\in S_\nu}\frac{1}{|Y|}\sum_{d\in D}\chi_C(d^{-1}x)\\ &= \frac{1}{|Y|}\sum_{d\in D}\frac{\left|C\cap d^{-1}S_\nu\right|}{|S_\nu|}\\ &\geq \frac{|D|}{|Y|}\frac{|C|}{|S_\nu|}-\frac{1}{|S_\nu|}\sum_{d\in D}\left|d^{-1}S_\nu\bigtriangleup S_\nu\right|\\ &\approx \frac{|C|}{|S_\nu|}\frac{|D|}{|Y|}.\end{aligned}$$

□

Theorem 2.2. *Suppose that G is a countable amenable group, $\mathcal{S}=(S_n)$ a Følner sequence for G, and $A,B\subseteq G$. If $d_{\mathcal{S}}(A)>\alpha$ and $\mathrm{BD}(B)>0$, then BA is $\mathcal{S}$-syndetic of level α.*

Proof. By [2, Lemma 3.2], there is (standard) $r>0$ and a finite subset T of G such that $d_{\mathcal{S}}(A)\cdot\mathrm{BD}(TB)>\alpha+r$. Since BA is $\mathcal{S}$-syndetic of level α if and only if TBA is $\mathcal{S}$-syndetic of level α, we may thus assume that $T=\{1\}$. Suppose that $\nu\in{}^*\mathbb{N}\backslash\mathbb{N}$ and $Y\subseteq{}^*G$ are obtained from A and B as in Lemma 2.1. Set $C={}^*A\cap S_\nu$ and $D={}^*B\cap Y$. Consider the internal set

$$\Gamma:=\left\{x\in S_\nu:\frac{\left|xC^{-1}\cap D\right|}{|Y|}\geq r\right\}.$$

Observe that

$$\begin{aligned}\alpha+r &< \frac{1}{|S_\nu|}\sum_{x\in S_\nu}\frac{\left|xC^{-1}\cap D\right|}{|Y|}\\ &= \frac{1}{|S_\nu|}\sum_{x\in\Gamma}\frac{\left|xC^{-1}\cap D\right|}{|Y|}+\frac{1}{|S_\nu|}\sum_{x\notin\Gamma}\frac{\left|xC^{-1}\cap D\right|}{|Y|}\leq\frac{|\Gamma|}{|S_\nu|}+r\end{aligned}$$

and hence $\mu_{S_\nu}(\Gamma)\geq\alpha$.

We now define by recursion a nested sequence of subsets (H_n) of G and a sequence (s_n) from G. First, set

$$H_0:=\left\{g\in G:\frac{1}{|\Gamma|}\left|\{x\in\Gamma:gx\notin{}^*(BA)\}\right|\not\approx 0\right\}.$$

Assuming H_n has been defined and is nonempty, let s_n be any element of H_n and set

$$H_{n+1}:=\left\{g\in G:\frac{1}{|\Gamma|}\left|\{x\in\Gamma:gx\notin{}^*(\{s_0,\ldots,s_n\}BA)\}\right|\not\approx 0\right\}.$$

If $H_n=\varnothing$ then we set $H_{n+1}=\varnothing$.

We first show that $H_n=\emptyset$ whenever $n>\left\lfloor\frac{1}{r}\right\rfloor-1$. Towards this end, suppose that $n\geq 1$ and $H_0,\ldots,H_{n-1}\neq\emptyset$. For $0\leq k\leq n-1$, fix $s_k\in H_k$ and $\gamma_k\in\Gamma$ such that

$$s_k\gamma_k\notin{}^*(\{s_0,\ldots,s_{k-1}\}BA).$$

Observe that the sets

$$s_0(\gamma_0C^{-1}\cap D),s_1(\gamma_1C^{-1}\cap D),\ldots,s_{n-1}(\gamma_{n-1}C^{-1}\cap D)$$

are pairwise disjoint. In fact, if

$$s_kD\cap s_j\gamma_jC^{-1}\neq\varnothing$$

for some $0 \leq k < j \leq n-1$, then,

$$s_j\gamma_j \in s_kDC \subseteq {}^*\left(\{s_0, \ldots, s_{j-1}\} BA\right),$$

contradicting our choice of γ_j. Recalling that Y is a Følner approximation for G, it follows that

$$1 \gtrsim \frac{1}{|Y|}\left|\bigcup_{k=0}^{n-1} s_k(\gamma_k C^{-1} \cap D)\right| = \sum_{k=0}^{n-1} \frac{\left|\gamma_k C^{-1} \cap D\right|}{|Y|} \geq nr$$

and hence $n \leq \frac{1}{r}$.

Take the least n such that $H_n = \varnothing$. (Note then that $n \leq \left\lfloor \frac{1}{r} \right\rfloor$, and that $n = 0$ is possible.) Set $E = \{1, s_0, \ldots, s_{n-1}\}$. (Note that $E = \{1\}$ if $n = 0$.) We claim that EBA is $\mathcal{S}$-thick of level α. Since $H_n = \varnothing$, for every $g \in G$, we have that

$$\frac{1}{|\Gamma|}\left|\{x \in \Gamma : gx \in {}^*(EBA)\}\right| \approx 1.$$

Therefore, for every finite subset L of G, we have that

$$\frac{1}{|\Gamma|}\left|\{x \in \Gamma : Lx \subseteq {}^*(EBA)\}\right| \approx 1.$$

Since $\mu_{S_\nu}(\Gamma) \geq \alpha$, we have that

$$\mu_{S_\nu}(\{x \in S_\nu \ : \ Lx \subseteq {}^*(EBA)\}) \geq \alpha.$$

It follows that

$$d_{\mathcal{S}}\left(\{x \in G : Lx \subseteq EBA\}\right) \geq \alpha. \quad \square$$

Lemma 2.3. *Suppose that $A \subseteq G$ is such that $d_{\mathcal{S}}(A \cup gA) = \alpha$ for all $g \in G$. Then A is $\mathcal{S}$-thick of level α.*

Proof. Take an infinite hyperfinite natural number ν so that $\mu_{S_\nu}({}^*A) = \alpha$. Fix $E \subseteq G$ finite and set

$$A_0 := \{x \in {}^*A \cap S_\nu \ : \ Ex \subseteq {}^*A\}.$$

It suffices to show that $\mu_{S_\nu}(A_0) = \alpha$. Set $R := ({}^*A \cap S_\nu) \setminus A_0$ and suppose, towards a contradiction, that $\mu_{S_\nu}(R) > 0$. Let $x \mapsto g_x : R \to E$ be an internal mapping such that $g_x x \notin {}^*A$. Since E is finite, there is $g \in E$ such that, setting $R_0 := \{x \in R \ : \ g_x = g\}$ (an internal set), we have $\mu_{S_\nu}(R_0) > 0$. It remains to observe that gR_0 is disjoint from ${}^*A$, whence by left invariance of μ_{S_ν}

$$d_{\mathcal{S}}(A \cup gA) \geq \mu_{S_\nu}({}^*A \cup gR_0) = \mu_{S_\nu}\left({}^*A\right) + \mu\left(gR_0\right) = \alpha + \mu_{S_\nu}\left(R_0\right) > \alpha,$$

contradicting our assumption on A. $\square$

Recall that the two-sided Banach density $\mathrm{BD}_2(B) > 0$ of B is defined as the supremum of $d_{\mathcal{S}}\left(B\right)$ when $\mathcal{S}$ ranges among all the *two-sided* Følner sequences for G.

Lemma 2.4. *If G is a countable amenable group and $B \subseteq G$ is such that $\mathrm{BD}_2(B) > 0$, then for every $\varepsilon > 0$ there exists $F \subseteq G$ finite such that $\mathrm{BD}_2(FB) > 1 - \varepsilon$.*

Proof. We proceed as in the "dynamical proof" of [2, Lemma 3.2]. In [3, Corollary 3.4], the authors prove a two-sided Furstenberg correspondence principle, a special case of which implies that there exists a compact metric space X, a continuous left action of G on X, an *ergodic* measure μ on X, and a clopen subset $\widehat{B}$ of X such that $\mathrm{BD}_2\left(B\right) = \mu(\widehat{B})$ and such that, for any $g_1, \ldots, g_n \in G$, we have

$$\mathrm{BD}_2\left(g_1B \cup \cdots \cup g_nB\right) \geq \mu((g_1 \cdot \widehat{B}) \cup \cdots \cup (g_n \cdot \widehat{B})).$$

(The corollary actually is in terms of intersections, not unions, but the proof readily adapts to the case of unions.) We therefore have:

$$\sup_F \mathrm{BD}_2(FB) \geq \sup_F \mu(F \cdot \widehat{B}) = \nu(G \cdot \widehat{B}).$$

Here, the suprema are taken over finite subsets of G. Since $G \cdot \widehat{B}$ is G-invariant and $\mu(\widehat{B}) > 0$, we have by the ergodicity of ν that $\mu(G \cdot \widehat{B}) = 1$, whence the lemma is proven. □

Theorem 2.5. *Suppose that G is a countable amenable group and $\mathcal{S}$ is a Følner sequence for G. If $A, B \subseteq G$ are such that $d_{\mathcal{S}}(A) = \alpha$ and $\mathrm{BD}_2(B) > 0$, then BA is $\mathcal{S}$-syndetic of level α.*

Proof. If A is $\mathcal{S}$-thick of level α then there is nothing to prove. If A is not $\mathcal{S}$-thick of level α, then by Proposition 2.3 there exists $g \in G$ such that $d_{\mathcal{S}}(A \cup gA) > \alpha$. By Lemma 2.4, after replacing B with FB for some finite set F, we may assume that B has 2-sided Banach density greater than $\frac{1}{2}$. Fix a two-sided Følner sequence $\mathcal{T} = (T_n)$ for G and $\nu > \mathbb{N}$ such that $\mu_{T_\nu}({}^*B) > \frac{1}{2}$. Since $\mathcal{T}$ is a two-sided Følner sequence, we have that $\mu_{T_\nu}({}^*(Bg^{-1})) = \mu_{T_\nu}({}^*B)$. It follows that $\mu_{T_\nu}({}^*(B \cap Bg^{-1})) > 0$, whence $B \cap Bg^{-1}$ has positive Banach density. By Theorem 2.2, the product of $B \cap Bg^{-1}$ and $A \cup gA$ is $\mathcal{S}$-syndetic of level α. It remains to observe that the product of $B \cap Bg^{-1}$ and $A \cup gA$ is contained in BA, and hence BA is $\mathcal{S}$-syndetic of level α as well. □

Observe that when G is abelian, and particularly when $G = \mathbb{Z}^d$, the Banach density and its two-sided analogue coincide. Therefore, as mentioned in the introduction, Theorem 14 and Theorem 18 of [4] are immediate consequences of Theorem 2.5, after observing that the sequence of sets $[-n, n]^d$ as well as any of its subsequences is a Følner sequence for $\mathbb{Z}^d$. Example 15 of [4] shows that the conclusion in Theorem 2.5 is optimal, even when G is the additive group of integers and $\mathcal{S}$ is the Følner sequence of intervals $[1, n]$.

3. A bound on the number of translates

The following theorem is a refinement of [5, Corollary 3.4]. In particular, we improve the bound on the number of translates, and also obtain an estimate on the $\mathcal{S}$-density of translates that witness the thickness of EBA.

Theorem 3.1. *Suppose that G is a countable amenable group, $\mathcal{S} = (S_n)$ a Følner sequence for G, and $A, B \subseteq G$. If $d_{\mathcal{S}}(A) \geq \alpha$ and $\mathrm{BD}(B) \geq \beta$, then there exists $s > 0$ and a finite subset $E \subseteq G$ such that $|E| \leq \frac{1}{\alpha\beta} - \frac{1}{\alpha} + 1$ and EBA is $\mathcal{S}$-thick of level s.*

Proof. Suppose that $Y \subseteq {}^*G$ and $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ are obtained from A and B as in Lemma 2.1. Set $C = {}^*A \cap S_\nu$ and $D = {}^*B \cap Y$. Consider the Loeb-measurable function $f : S_\nu \to [0, 1]$ defined by

$$f(x) = \mu_Y(xC^{-1} \cap D) = \mathrm{st}\left(\frac{|xC^{-1} \cap D|}{|Y|}\right).$$

A simple approximation argument shows that

$$\int f d_{\mu_{S_\nu}} = \mathrm{st}\left(\frac{1}{|S_\nu|} \sum_{x \in S_\nu} \frac{|xC^{-1} \cap D|}{|Y|}\right);$$

see also [1, Theorem 3.2.9].

Since

(1) $f \geq \int f d_{\mu_{S_\nu}}$ on a positive measure set,
(2) any Loeb measurable set is approximated arbitrarily well from within by an internal set, and
(3) any internal function attains its minimum value on a hyperfinite set,

we deduce that there is a standard $r>0$ and an infinitesimal $\eta>0$, so that, setting

$$\Gamma := \left\{ x \in S_\nu : \frac{1}{|Y|} \left| xC^{-1} \cap D \right| \geq \alpha\beta - \eta \right\},$$

we have $|\Gamma| > r\,|S_\nu|$.

Fix a family $(p_g)_{g\in G}$ of strictly positive standard real numbers such that $\sum_{g\in G} p_g \leq \frac{1}{2}$. We now define a sequence of subsets (H_n) of G and a sequence (s_n) from G. Define

$$H_0 := \left\{ g \in G : \frac{1}{|\Gamma|} \left| \{ x \in \Gamma : gx \notin {}^*(BA) \} \right| > p_g \right\}.$$

If H_n has been defined and is nonempty, let s_n be any element of H_n and set

$$H_{n+1} := \left\{ g \in G : \frac{1}{|\Gamma|} \left| \{ x \in \Gamma : gx \notin {}^*(\{s_0, \ldots, s_n\} BA) \} \right| > p_g \right\}.$$

If $H_n = \varnothing$ then we set $H_{n+1} = \varnothing$.

We claim $H_n = \emptyset$ for $n > \left\lfloor \frac{1}{\alpha\beta} - \frac{1}{\alpha} \right\rfloor$. Towards this end, suppose $H_n \neq \varnothing$. For $0 \leq k \leq n$, take $\gamma_k \in \Gamma$ such that

$$s_k\gamma_k \notin {}^*(\{s_0, \ldots, s_{k-1}\} BA).$$

Observe that the sets s_0D, $s_1(\gamma_1C^{-1} \cap D)$, ..., $s_n(\gamma_nC^{-1} \cap D)$ are pairwise disjoint. In fact, if

$$s_iD \cap s_j\gamma_jC^{-1} \neq \varnothing$$

for $0 \leq i < j \leq n$, then

$$s_j\gamma_j \in s_iDC \subseteq {}^*(\{s_0, \ldots, s_{j-1}\} BA),$$

contradicting the choice of γ_j. Therefore we have that

$$\begin{aligned}
1 &\gtrsim \frac{1}{|Y|} \left| s_0D \cup s_1\left((\gamma_1C^{-1} \cap D\right) \cup \cdots \cup s_n\left((\gamma_nC^{-1} \cap D\right) \right| \\
&\geq \frac{1}{|Y|} \left(|D| + \sum_{i=1}^{n} \left| \gamma_iC^{-1} \cap D \right| \right) \\
&\geq \beta + \alpha\beta n.
\end{aligned}$$

It follows that $n \leq \left\lfloor \frac{1}{\alpha\beta} - \frac{1}{\alpha} \right\rfloor$.

Take the least n such that $H_n = \emptyset$. Note that $n \leq \left\lfloor \frac{1}{\alpha\beta} - \frac{1}{\alpha} \right\rfloor + 1$. If $n=0$ then BA is already $\mathcal{S}$-thick of level r, and there is nothing to prove. Let us assume that $n \geq 1$, and set $E = \{s_0, \ldots, s_{n-1}\}$. It follows that, for every $g \in G$, we have that

$$\frac{1}{|\Gamma|} \left| \{ x \in \Gamma : gx \in {}^*(EBA) \} \right| \geq 1 - p_g.$$

Suppose that L is a finite subset of G. Then

$$\frac{1}{|\Gamma|} \left| \{ x \in \Gamma : Lx \subseteq {}^*(EBA) \} \right| \geq 1 - \sum_{g\in L} p_g \geq 1 - \sum_{g\in G} p_g \geq \frac{1}{2}.$$

Therefore

$$\frac{1}{|S_\nu|} \left| \{ x \in S_\nu : Lx \subseteq {}^*(EBA) \} \right| \geq \frac{r}{2}.$$

This shows that EBA is $\mathcal{S}$-thick of level $\frac{r}{2}$. □

With a similar argument and using Markov's inequality [9, Lemma 1.3.15] one can also prove the following result. We omit the details.

Theorem 3.2. *Suppose that G is a countable amenable group, $\mathcal{S} = (S_n)$ a Følner sequence for G, and $A, B \subseteq G$. If $d_{\mathcal{S}}(A) \geq \alpha$ and $\mathrm{BD}(B) \geq \beta$, then for every $\gamma \in (0, \alpha\beta]$ there exists $E \subseteq G$ such that $|E| \leq \frac{1-\beta}{\gamma} + 1$ and EBA is $\mathcal{S}$-syndetic of level $\frac{\alpha\beta-\gamma}{1-\gamma}$.*

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